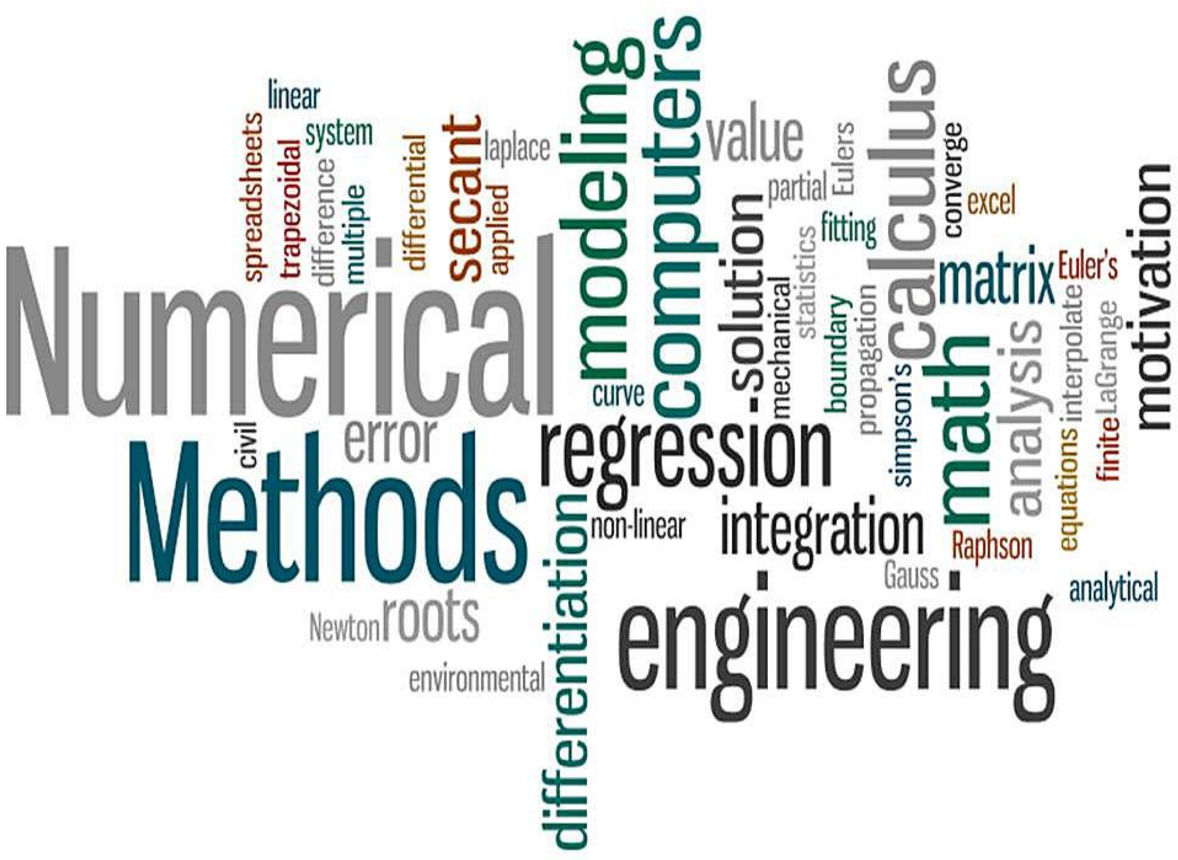
جامعة المنارة
MANARA UNIVERSITY

Numerical Analysis Programming



Roots Of Equations
Part-02

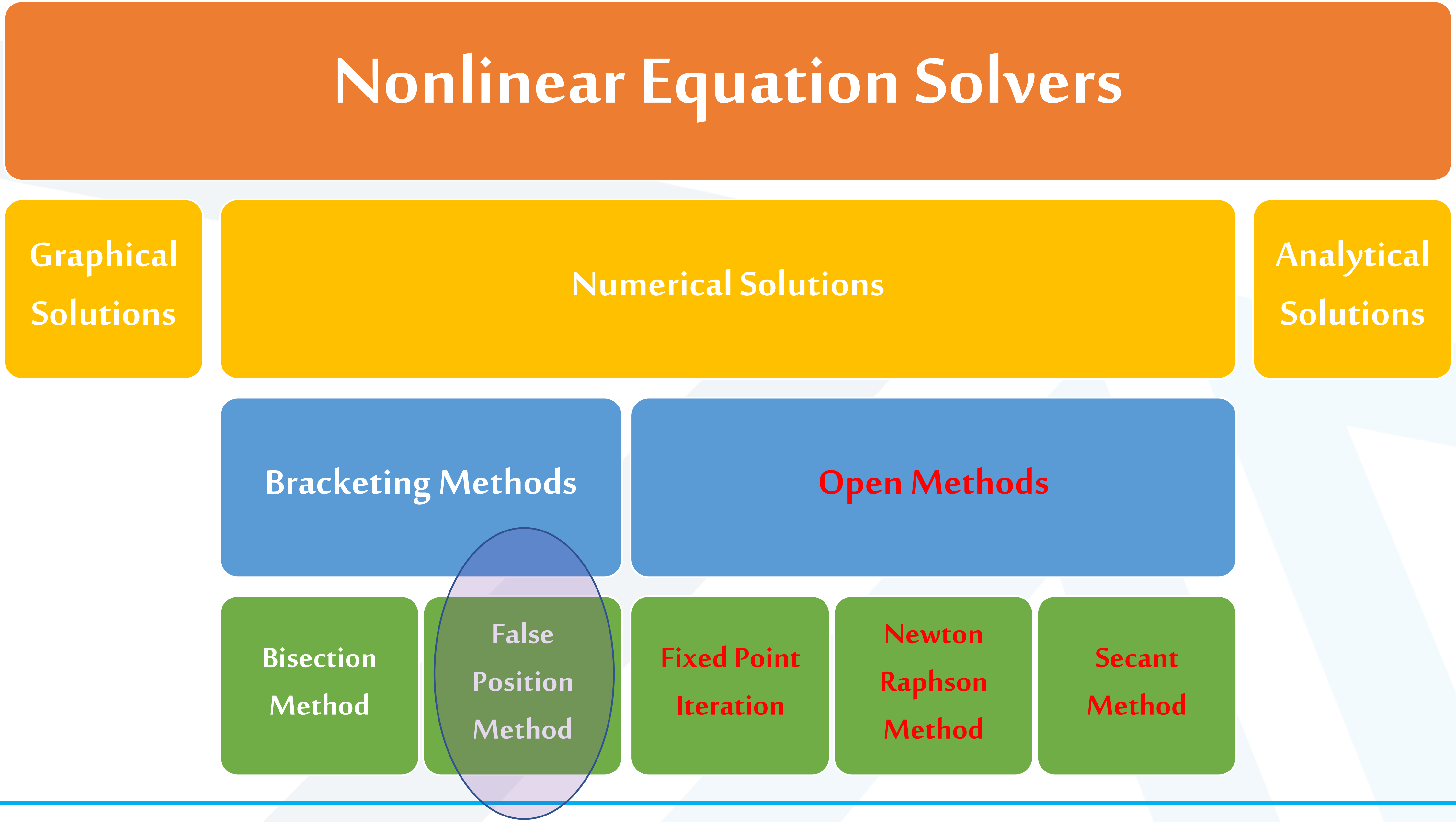
Root Finding Problems

Many problems in Science and Engineering are expressed as:

Given a continuous function $f(x)$,

find the value r such that $f(r) = 0$

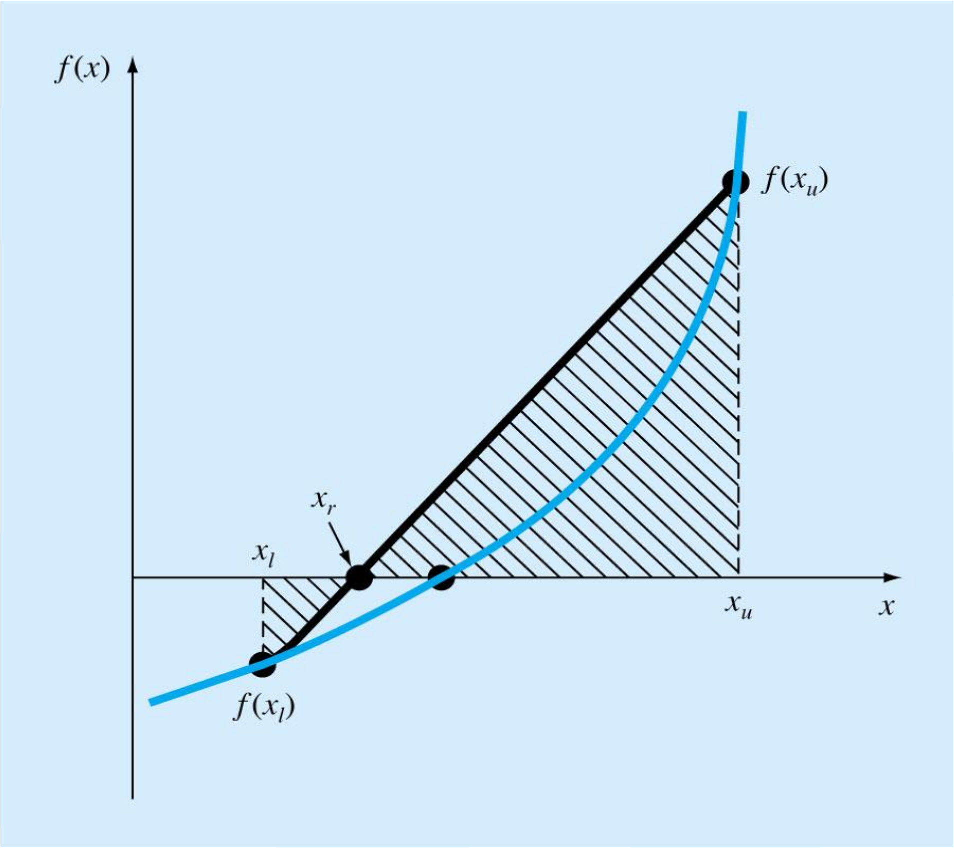
These problems are called root finding problems.


Nonlinear Equation Solvers
Graphical Solutions
Numerical Solutions
Analytical Solutions
Bracketing Methods
Open Methods
Bisection Method
False Position Method
Fixed Point Iteration
Newton Raphson Method
Secant Method

False Position Method

- If a real root is bounded by xl and xu of f(x)=0, then we can approximate the solution by doing a linear interpolation between the points [xl, f(xl)] and [xu, f(xu)] to find the xr value such that l(xr)=0, l(x) is the linear approximation of f(x).

$$x_r = \frac{x_l f_u - x_u f_l}{f_u - f_l}$$

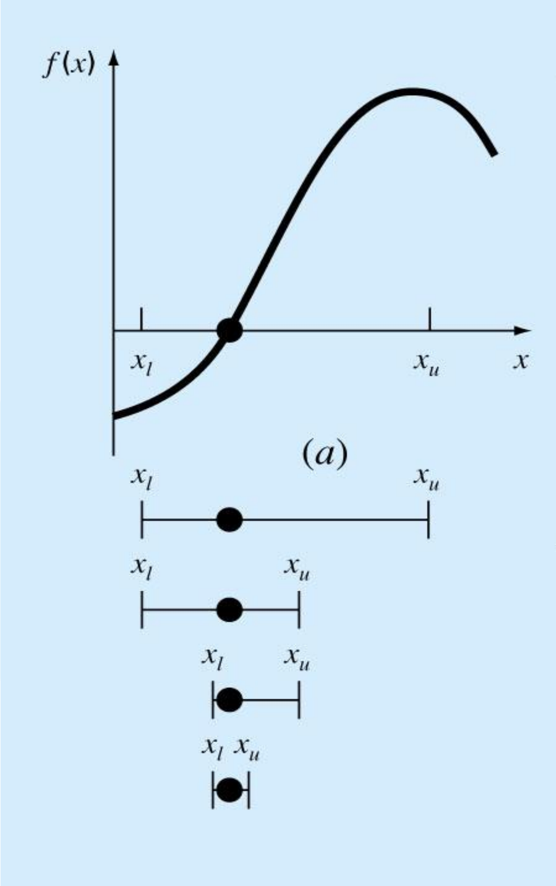


Nonlinear Equation Solvers

- Graphical Solutions
- Numerical Solutions
  - Bracketing Methods
    - Bisection Method
    - False Position Method
  - Open Methods
    - Fixed Point Iteration
    - Newton Raphson Method
    - Secant Method
- Analytical Solutions

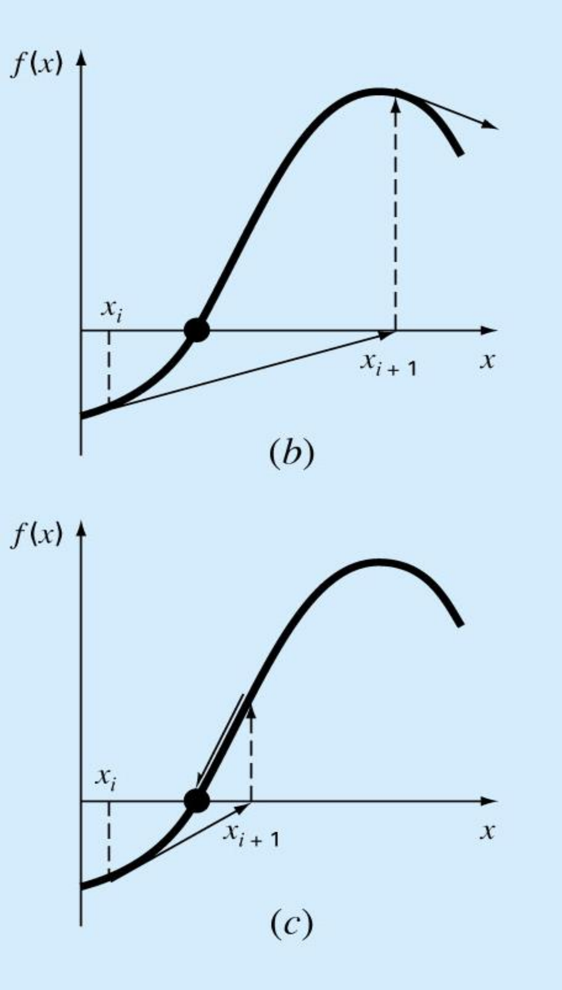
Open Methods

- For the **bracketing methods** which is discussed in previous lecture, the root is located within an interval prescribed by a lower and an upper bound. Repeated application of these methods always results in closer estimates of the true value of the root. Such methods are said to be convergent because they move closer to the truth as the computation progresses.



Open Methods

- In contrast, the **open methods** described in this lecture are based on formulas necessarily bracket the root. As such, they sometimes diverge or move away from the true root as the computation progresses (Fig. b). However, when the open methods converge (Fig. c), they usually do so much more quickly than the bracketing methods.
- We will begin our discussion of open techniques with a simple version that is useful for illustrating their general form and also for demonstrating the concept of convergence.



Simple Fixed Point Iteration

- Also known as **one-point iteration** or **successive substitution**
- To find the root for $f(x) = 0$, we reformulate $f(x) = 0$ so that there is an x on one side of the equation.

$$f(x) = 0 \quad \Leftrightarrow \quad g(x) = x$$

- If we can solve $g(x) = x$, we solve $f(x) = 0$.
  - x is known as the fixed point of $g(x)$.
- We solve $g(x) = x$ by computing

$$x_{i+1} = g(x_i) \quad \text{with } x_0 \text{ given}$$

until x_{i+1} converges to x.

Simple Fixed Point Iteration

$$f(x) = x^2 + 2x - 3 = 0$$

$$x^2 + 2x - 3 = 0 \quad \Rightarrow \quad 2x = 3 - x^2 \quad \Rightarrow \quad x = \frac{3 - x^2}{2}$$

$$\Rightarrow x_{i+1} = g(x_i) = \frac{3 - x_i^2}{2}$$

Reason: If x converges, i.e. $x_{i+1} \rightarrow x_i$

$$x_{i+1} = \frac{3 - x_i^2}{2} \rightarrow x_i = \frac{3 - x_i^2}{2}$$

$$\Rightarrow x_i^2 + 2x_i - 3 = 0$$

$\sin x = 0$??????????

Simple Fixed Point Iteration

- **Example :** Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.
- **Solution:** The function can be separated directly and expressed in Equation as: $x_{i+1} = e^{-x_i}$. Starting with an initial guess of x_i =0, the iterative equation can be applied to compute:

i	x_i	ε_a (%)	ε_t (%)
0	0		100.0
1	1.000000	100.0	76.3
2	0.367879	171.8	35.1
3	0.692201	46.9	22.1
4	0.500473	38.3	11.8
5	0.606244	17.4	6.89
6	0.545396	11.2	3.83
7	0.579612	5.90	2.20
8	0.560115	3.48	1.24
9	0.571143	1.93	0.705
10	0.564879	1.11	0.399

Thus, each iteration brings the estimate closer to the true value of the root: 0.56714329.

Newton Raphson Method

- Given an initial guess of the root x_0, Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.
- **Based on Taylor series expansion:**

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + f''(x_i)\frac{\Delta x^2}{2!} + O\Delta x^3$$

The root is the value of x_{i+1} when $f(x_{i+1}) = 0$

Rearranging,

Solve for

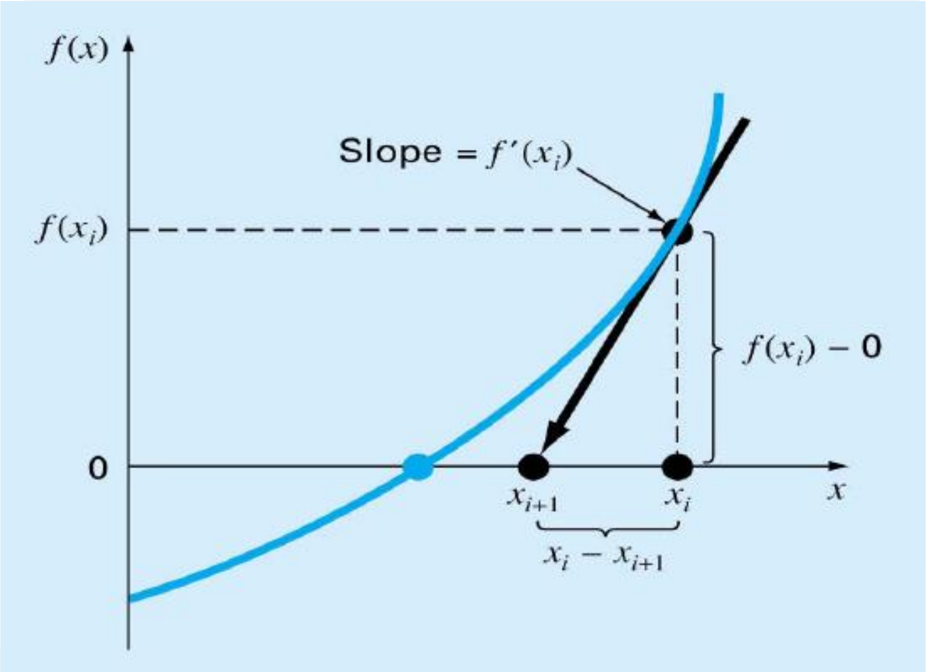
$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Newton-Raphson formula

Newton Raphson Method

- **Graphical Depiction:** If the initial guess at the root is x_i, then a tangent to the function of x_i that is $f'(x_i)$ is extrapolated down to the x-axis to provide an estimate of the root at x_{i+1}.



A convenient method for functions whose derivatives can be evaluated analytically.

Newton Raphson Method

- **Example:** Use the Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$, employing an initial guess of $x_0 = 0$
- **Solution:** The first derivative of the function can be evaluated as: $f'(x) = -e^{-x} - 1$ which can be substituted along with the original function: $x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{e^{-x_i} - 1}$

  Starting with an initial guess of $x_0 = 0$, the iterative equation can be applied to compute:

Pitfalls of Newton Raphson Method

$$Assumptions: \quad f(x),\ f'(x),\ x_0 \quad are\ available,$$

$$f'(x_0) \neq 0$$

$$Newton's\ Method\ new\ estimate:$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Problem :

$f'(x_i)$ is not available,

or difficult to obtain analytically.

It may not be convenient for functions whose derivatives cannot be evaluated analytically.

The Secant Method

- A slight variation of Newton's method for functions whose **derivatives are difficult to evaluate**. For these cases the derivative can be approximated by a backward finite divided difference.

The Secant Method - Derivation

The secant method can also be derived from geometry:

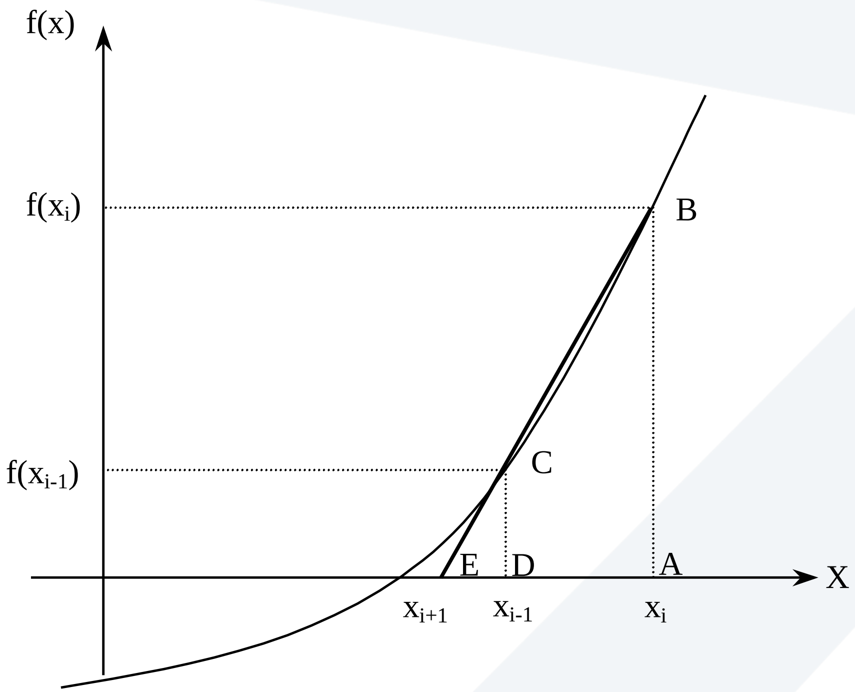


Figure 2 Geometrical representation of the Secant method.

The Geometric Similar Triangles

$$\frac{AB}{AE} = \frac{DC}{DE}$$

can be written as

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

The Secant Method

- **Example:** Use the secant method to estimate the root of $f(x) = e^{-x} - x$, start with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$
- **Solution:**

Solution. Recall that the true root is 0.56714329. . . .

First iteration:

$$x_{-1} = 0 \qquad f(x_{-1}) = 1.00000$$

$$x_0 = 1 \qquad f(x_0) = -0.63212$$

$$x_1 = 1 - \frac{-0.63212(0-1)}{1-(-0.63212)} = 0.61270 \qquad \varepsilon_t = 8.0\%$$

Second iteration:

$$x_0 = 1 \qquad f(x_0) = -0.63212$$

$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

(Note that both estimates are now on the same side of the root.)

$$x_2 = 0.61270 - \frac{-0.07081(1-0.61270)}{-0.63212-(-0.07081)} = 0.56384 \qquad \varepsilon_t = 0.58\%$$

Third iteration:

$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

$$x_2 = 0.56384 \qquad f(x_2) = 0.00518$$

$$x_3 = 0.56384 - \frac{0.00518(0.61270-0.56384)}{-0.07081-(-0.00518)} = 0.56717 \qquad \varepsilon_t = 0.0048\%$$

Homework

Problem Statement: Determine the highest real root of:

$$f(x) = -6 + 17.5x - 11.6x^2 + 2.1x^3$$

a) Graphically.

b) Fixed Point iteration method (three iterations, $x_0=3$).

c) Newton Raphson method (three iterations, $x_0=3$).

d) Secant method (three iterations, $x_{-1}=3$, $x_0=4$).

Compute the approximate percent relative errors for your solutions.